Discrete Mathematics 340 (2017) 927–934

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Maximum number of edges in claw-free graphs whose maximum degree and matching number are bounded

Cemil Dibek [a], Tınaz Ekim [a,*], Pinar Heggernes [b]

[a] Department of Industrial Engineering, Bogazici University, Istanbul, Turkey
[b] Department of Informatics, University of Bergen, Norway

ARTICLE INFO

Article history:
Received 29 July 2016
Received in revised form 12 January 2017
Accepted 12 January 2017

Keywords:
Edge-extremal graphs
Claw-free graphs
Maximum matching
Ramsey theory

ABSTRACT

We determine the maximum number of edges that a claw-free graph can have, when its maximum degree and matching number are bounded. This is a famous problem that has been studied on general graphs, and for which there is a tight bound. The graphs achieving this bound contain in most cases an induced copy of $K_{1,3}$, the claw, which motivates studying the question on claw-free graphs. Note that on general graphs, if one of the mentioned parameters is not bounded, then there is no upper bound on the number of edges. We show that on claw-free graphs, bounding the matching number is sufficient for obtaining an upper bound on the number of edges. The same is not true for the degree, as a long path is claw-free. We give exact tight formulas for both when only the matching number is bounded and when both parameters are bounded. We also construct claw-free graphs whose edge numbers match the given bounds.

1. Introduction

Extremal graph theory is an important field in discrete mathematics. It studies questions like how large or small some parameter of a graph can be under a given set of conditions [2,9]. An important problem in this respect is the following: Given a family $\mathcal{L}$ of forbidden subgraphs, find those graphs which contain no graph in $\mathcal{L}$ as a subgraph and have the maximum number of edges. In 1941, Turán [11] described the so-called Turán graphs on n vertices that do not contain the complete graph K_k, for a fixed $k < n$, and have the maximum number of edges. The Erdős–Stone theorem from 1946 [6] extends Turán's result by bounding the number of edges in a graph that does not have a fixed Turán graph as a subgraph.

A similar extremal question that dates back to 1960 is the following: *What is the maximum number of edges that a graph can have if its maximum degree is less than i and the size of a maximum matching of it is less than j, for two given integers i and j?* This question is a special case of a more general problem studied by Erdős and Rado [5], and it was first resolved by Chvátal and Hanson [4]. Later, Balachandran and Khare [1] gave a more structural proof and they also identified some graphs whose number of edges matches the given upper bound. In most cases, these graphs have connected components each of which is a star, i.e., $K_{1,k}$ for an appropriate integer k. This gives rise to the question what happens if the smallest star,[1] $K_{1,3}$, is forbidden as an induced subgraph. This restriction describes exactly the class of claw-free graphs.

In this paper we give an exact formula for the maximum number of edges that a claw-free graph can have, when its maximum degree and matching number are bounded. First we show that bounding only the matching number of a claw-free graph already results in bounded number of edges, and we give a formula for this number. For large enough given bound

* Corresponding author.
E-mail addresses: cemil.dibek@boun.edu.tr (C. Dibek), tinaz.ekim@boun.edu.tr (T. Ekim), pinar.heggernes@uib.no (P. Heggernes).

[1] Here we take as the smallest star, a star that is not also a path.

http://dx.doi.org/10.1016/j.disc.2017.01.010

on the maximum degree, this number is smaller than that for general graphs, and the gap increases with increasing allowed degree. If the degree bound is small enough, then we are able to give a formula that matches the formula for general graphs. In all cases, we also describe the claw-free graphs whose edge numbers match the given upper bounds.

2. Preliminaries

We work with undirected and simple graphs. Such a graph is denoted by $G = (V, E)$, where V is the set of vertices and E is the set of edges of G, with $|V| = n$. The *neighborhood* of a vertex is the set of all vertices adjacent to it. The maximum degree of a vertex in G is denoted by $\Delta(G)$. The size of a maximum matching in G is called its *matching number* and denoted by $\nu(G)$. G has a *perfect matching* if $\nu(G) = n/2$. A set X of vertices is an *independent set* if no pair of vertices in X are adjacent, whereas X is a *clique* if every pair of vertices in X are adjacent. A vertex in a connected graph is a *cut vertex* if removing it disconnects the graph.

The complete bipartite graph $K_{1,i-1}$ is called an *i-star*, and a 4-star ($K_{1,3}$) is called a *claw*. A vertex whose neighborhood contains an independent set of size 3 is called a *claw-center*. A graph is *claw-free* if it does not have a claw as an induced subgraph. Equivalently, a graph is claw-free if none of its vertices is a claw-center. The following observation is thus obvious; it will be used frequently in our proofs.

Observation 1. *If the neighborhood of a vertex v can be partitioned into two cliques, then v is not a claw-center.*

For a given graph class $\mathcal{C}$ and two given positive integers i and j, we define $\mathcal{M}_{\mathcal{C}}(i, j)$ to be the set of all graphs G in $\mathcal{C}$ satisfying $\Delta(G) < i$ and $\nu(G) < j$. A graph in $\mathcal{M}_{\mathcal{C}}(i, j)$ with the maximum number of edges is called *edge-extremal*. Thus the question that we are resolving in this paper is determining the number of edges in edge-extremal claw-free graphs. In the remainder, we assume that edge-extremal graphs have no isolated vertices since adding isolated vertices to a graph does not increase the number of edges. We let $\mathcal{GEN}$ denote the class of all graphs, and $\mathcal{CF}$ the class of claw-free graphs.

The following theorem summarizes the results of Balachandran and Khare [1]. For a positive integer i, they defined K_i' to be the graph obtained by removing a perfect matching from the complete graph K_i on i vertices, adding a new vertex v, and making v adjacent to $i - 1$ of the other vertices.

Theorem 2 ([1]). *The maximum number of edges in an edge-extremal graph in $\mathcal{M}_{\mathcal{GEN}}(i, j)$ is*

$$(i-1)(j-1) + \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{j-1}{\lceil \frac{i-1}{2} \rceil} \right\rfloor .$$

A graph with this number of edges is obtained by taking the disjoint union of r copies of i-star and q copies of

$$\begin{cases} K_i & \text{if } i \text{ is odd} \\ K_i' & \text{if } i \text{ is even}, \end{cases}$$

where q is the largest integer such that $j - 1 = q \left\lceil \frac{i-1}{2} \right\rceil + r$ and $r \geq 0$.

In the next two sections we will give the corresponding numbers and extremal graphs for $\mathcal{M}_{\mathcal{CF}}(i, j)$. First, in the following section we show that bounding only the matching number is sufficient to be able to bound the number of edges of a claw-free graph.

3. Maximum number of edges in claw-free graphs whose matching number is bounded

We start with the following lemma, which does not hold for arbitrary graphs, in particular for stars.

Lemma 3. *For a connected claw-free graph G, $n \leq 2\nu(G) + 1$.*

Proof. Let G be a connected claw-free graph. Sumner [10] has shown that every connected claw-free graph with an even number of vertices has a perfect matching. Consequently, if n is even, then $\nu(G) = \frac{n}{2}$, and the result follows. If n is odd, then remove a vertex from G that is not a cut vertex. Such a vertex exists by the results of Chartrand and Zhang [3]. The remaining graph is connected, claw-free, and has an even number of vertices; thus it has a perfect matching whose size is $(n-1)/2$. Consequently, $\nu(G) \geq (n-1)/2$, and the proof is complete. □

We now describe the edge-extremal claw-free graphs when the given bound on the maximum degree is large enough.

Theorem 4. *If $i \geq 2j$, then K_{2j-1} is the unique edge-extremal graph in $\mathcal{M}_{\mathcal{CF}}(i, j)$, resulting in $(2j-1)(j-1)$ edges.*

Proof. Since K_{2j-1} is claw-free, $\nu(K_{2j-1}) = j - 1 < j$, and $\Delta(K_{2j-1}) = 2j - 2 \leq i - 2 < i$, we have that $K_{2j-1} \in \mathcal{M}_{\mathcal{CF}}(i, j)$. Let us show that K_{2j-1} is the edge-extremal graph in $\mathcal{M}_{\mathcal{CF}}(i, j)$. When $i \geq 2j$, if an edge-extremal graph in $\mathcal{M}_{\mathcal{CF}}(i, j)$ is connected, then by Lemma 3, it can have at most $2j - 1$ vertices. The graph that obtains the maximum number of edges on a given set of vertices is unique; it is the complete graph K_{2j-1}, and it has $(2j-1)(j-1)$ edges. Now assume that, for $i \geq 2j$, there is a disconnected edge-extremal graph G in $\mathcal{M}_{\mathcal{CF}}(i,j)$ with $k \geq 2$ connected components $G_1, G_2, \ldots, G_k$. Let the matching number of G_ℓ be j_ℓ for $1 \leq \ell \leq k$, such that $j_\ell \geq 1$ (recall that G has no isolated vertices) and $\sum_{\ell=1}^{k} j_\ell = j - 1$. By Lemma 3, each G_ℓ has at most $2j_\ell + 1$ vertices, i.e. each G_ℓ has at most $\frac{(2j_\ell+1)2j_\ell}{2} = (2j_\ell + 1)j_\ell$ edges. Then the maximum number of edges of G is given by:

$$\sum_{\ell=1}^{k}(2j_\ell + 1)j_\ell = \sum_{\ell=1}^{k} j_\ell + 2\sum_{\ell=1}^{k} j_\ell^2 < \sum_{\ell=1}^{k} j_\ell + 2\left(\sum_{\ell=1}^{k} j_\ell\right)^2 = j - 1 + 2(j-1)^2 = (2j-1)(j-1).$$

This contradicts that G is edge-extremal as it has fewer edges than K_{2j-1}. We conclude that K_{2j-1} is the unique edge-extremal graph in $\mathcal{M}_{\mathcal{CF}}(i,j)$. □

Consequently, if there is no bound on i, then j limits the number of edges in a claw-free graph, and we have the following corollary.

Corollary 5. *A claw-free graph whose matching number is less than j has at most $(2j-1)(j-1)$ edges and it is isomorphic to K_{2j-1}.*

As we formalize in the next corollary, if $i = 2j$, then edge-extremal graphs in $\mathcal{M}_{\mathcal{GEN}}(i,j)$ and in $\mathcal{M}_{\mathcal{CF}}(i,j)$ have the same number of edges. However, as the difference $i - 2j$ gets larger, the difference in the number of edges between the edge-extremal graphs in $\mathcal{M}_{\mathcal{GEN}}(i,j)$ and in $\mathcal{M}_{\mathcal{CF}}(i,j)$ gets larger.

Corollary 6. *If $i \geq 2j$, then the difference in the number of edges between edge-extremal graphs in $\mathcal{M}_{\mathcal{GEN}}(i,j)$ and in $\mathcal{M}_{\mathcal{CF}}(i,j)$ is $(j-1)(i-2j)$.*

Proof. If $i \geq 2j$, then $\lceil \frac{i-1}{2} \rceil > j - 1$. This gives $q = 0$ and $r = j - 1$ in the statement of Theorem 2. Applying Theorem 2 with these values, we get that an edge-extremal graph in $\mathcal{M}_{\mathcal{GEN}}(i,j)$ has $(i-1)(j-1)$ edges. This can be obtained by $j-1$ copies of i-star. By Theorem 4, the difference in the number of edges is $(i-1)(j-1) - (2j-1)(j-1) = (j-1)(i-2j)$. □

4. Edge-extremal claw-free graphs

If $i < 2j$ then the result of the previous section cannot be applied to edge-extremal graphs in $\mathcal{M}_{\mathcal{CF}}(i,j)$, as K_{2j-1} has too high degree and does not belong to $\mathcal{M}_{\mathcal{CF}}(i,j)$ in this case. Interestingly, we will prove that the number of edges in edge-extremal graphs of $\mathcal{M}_{\mathcal{GEN}}(i,j)$ and in those of $\mathcal{M}_{\mathcal{CF}}(i,j)$ is the same in this case. Our proofs are constructive; we describe in detail such edge-extremal claw-free graphs. These are completely different from the graphs described by Balachandran and Khare [1] for general edge-extremal graphs.

4.1. Elementary cases

Let us start by identifying some elementary cases; in particular the cases $r = 0$, $i = 2$, $i = 3$, and $j = 2$, where r is the remainder of dividing $j - 1$ by $\lceil \frac{i-1}{2} \rceil$. These cases are all covered by the coming proofs; they are merely given to provide more insight at this point.

• $r = 0$

It is easy to observe that K_i' is claw-free for all i. Hence, when $r = 0$, we can immediately conclude from Theorem 2 that the described edge-extremal graphs in $\mathcal{M}_{\mathcal{GEN}}(i,j)$ are also edge-extremal graphs in $\mathcal{M}_{\mathcal{CF}}(i,j)$. In fact, Balachandran and Khare [1] state that when $r = 0$ then every edge-extremal graph has $\frac{j-1}{\lceil \frac{i-1}{2} \rceil}$ components, and $\lceil \frac{i-1}{2} \rceil$ is the matching number of each of these components. Thus the edge-extremal graph is unique in this case.

• $i \in \{2, 3\}$

Theorem 2 implies that when $i = 2$, an edge-extremal graph in $\mathcal{M}_{\mathcal{GEN}}(i,j)$ has $j - 1$ edges. Such an edge-extremal graph can simply be obtained by the disjoint union of $j-1$ copies of K_2, which is claw-free. When $i = 3$, an edge-extremal graph in $\mathcal{M}_{\mathcal{GEN}}(i,j)$ has $3(j-1)$ edges. Such a graph can only be obtained by the disjoint union of $j-1$ copies of K_3, which is claw-free.

• $j = 2$

In this case, Theorem 2 implies that the unique edge-extremal graph is an i-star, which has $i - 1$ edges. However this graph is clearly not claw-free, and hence Theorem 2 is not applicable in this case. On the other hand, since connected claw-free graphs on an even number of vertices have perfect matchings, and we excluded isolated vertices from edge-extremal graphs, we can conclude that the unique claw-free graph with matching number 1 that has the maximum number of edges is K_3.

4.2. Construction of some special claw-free graphs

In this section we give constructions of two special types of claw-free graphs, which will be used to prove our main result.

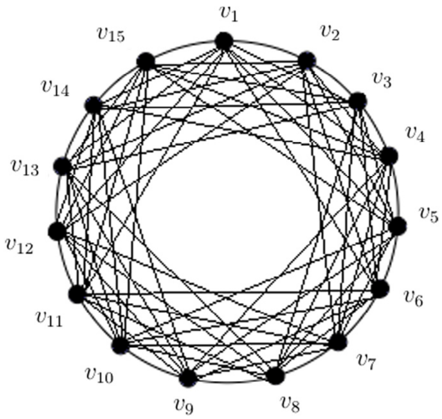

Fig. 1. The graph $R_{15,10}$.

Given two integers p and r, where r is even, first we construct a particular graph that we call $R_{p,r}$. This graph has p vertices, and each vertex has degree r. Let $r = 2k$ where $k \in \mathbb{Z}^+$. We construct $R_{p,r}$ as follows: the vertex set of $R_{p,r}$ is $\{v_i : i \in \{1, 2, \ldots., p\}\}$, and every v_i is adjacent to $v_{i-k}, v_{i-k+1}, \ldots., v_{i-1}, v_{i+1}, \ldots., v_{i+k-1}, v_{i+k}$, where indices are of modulo p. As an example, $R_{15,10}$ is given in Fig. 1. Observe that the neighborhood of every vertex in $R_{p,r}$ can be partitioned into two cliques. Thus by Observation 1, it follows that $R_{p,r}$ is claw-free.

Next we define another type of claw-free graphs. Given p and r that are both odd, such that $r + 2 \leq p \leq 2r + 1$, we construct a particular graph that we call $R'_{p,r}$. This graph has p vertices; $p - 1$ of these vertices have degree r, and one of them has degree $r - 1$. In order to construct $R'_{p,r}$, we first construct $R_{p,r-1}$, and then we add some additional edges in a systematic way. The vertex set of $R'_{p,r}$ is $\{v_i : i \in \{1, 2, \ldots., p\}\}$. The edge set is given by $E \cup E'$, where E is the edge set of $R_{p,r-1}$, and E' is a set of $(p-1)/2$ edges that we specify below.

Let m be the greatest common divisor of p and $r+1$; we denote this as $\gcd(p, r+1) = m$. For each ℓ such that $1 \leq \ell \leq m$, we define the sequence $s_\ell = \{\ell + t(\frac{r+1}{2}) \,(mod\ p) \mid 0 \leq t \leq \frac{p}{m} - 1\}$. In other words, we generate the following sequences, where the elements of these sequences are all in modulo p:

$$s_1 = \left\{1, \left(1 + \left(\frac{r+1}{2}\right)\right), \left(1 + 2\left(\frac{r+1}{2}\right)\right), \left(1 + 3\left(\frac{r+1}{2}\right)\right), \ldots\ldots.., \left(1 + \left(\frac{p}{m} - 1\right)\left(\frac{r+1}{2}\right)\right)\right\}$$
$$s_2 = \left\{2, \left(2 + \left(\frac{r+1}{2}\right)\right), \left(2 + 2\left(\frac{r+1}{2}\right)\right), \left(2 + 3\left(\frac{r+1}{2}\right)\right), \ldots\ldots.., \left(2 + \left(\frac{p}{m} - 1\right)\left(\frac{r+1}{2}\right)\right)\right\}$$
$$\vdots$$
$$s_m = \left\{m, \left(m + \left(\frac{r+1}{2}\right)\right), \left(m + 2\left(\frac{r+1}{2}\right)\right), \left(m + 3\left(\frac{r+1}{2}\right)\right), \ldots\ldots.., \left(m + \left(\frac{p}{m} - 1\right)\left(\frac{r+1}{2}\right)\right)\right\}.$$

Let us point out the following important properties of these sequences.

- *There are in total p elements.* Each sequence has p/m elements because $0 \leq t \leq p/m - 1$. Since there are m sequences, there are in total p elements.
- *The elements in a sequence are equivalent modulo m.* This follows from the fact that $(r + 1)/2$ is a multiple of m. To see this fact, recall first that $\gcd(p, r+1) = m$. Observe that m is odd since p is odd. If $r + 1 = mk$ for some k, then k must be even since $r + 1$ is even and m is odd. If $k = 2a$, then $r + 1 = 2ma$, and $(r + 1)/2 = ma$, and thus a multiple of m. Given this, and the fact that in a sequence in order to get the next element, we only add $(r + 1)/2$, we conclude that the elements in a sequence are equivalent modulo m.
- *The sequences are disjoint, i.e., $s_i \cap s_j = \emptyset$ for $i \neq j$ and $i, j \in \{1, 2, \ldots, m\}$.* Seeing that elements in a sequence are equivalent modulo m, and also that each sequence s_ℓ starts with number ℓ where $1 \leq \ell \leq m$, we deduce that two elements from two different sequences cannot be the same because the first elements of the sequences are not equivalent modulo m.

Since the elements in the sequences are different and there are in total p elements all written in modulo p, we conclude that the union of all elements is exactly the set of integers from 0 to $p - 1$. Let us replace 0 by p, so that we get integers from 1 to p instead. These are exactly the indices of the vertices of our graph. The order of the elements in each sequence is crucial. This order will determine between which pairs of vertices we will add an edge. We form a long sequence from the

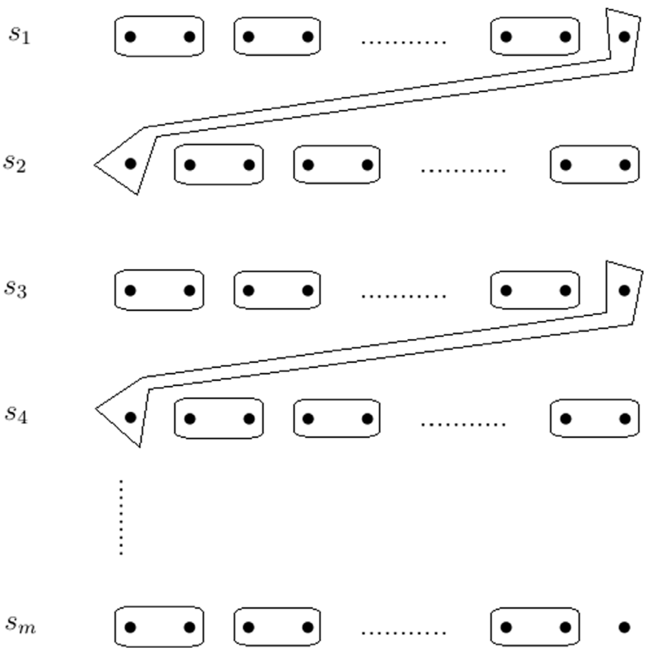

Fig. 2. The description of the edges in E'.

sequences s_ℓ for $1 \le \ell \le m$, by concatenating these sequences in the following way:

$$\left\{ \underbrace{1, \ldots\ldots, \left(1 + \left(\frac{p}{m} - 1\right)\left(\frac{r+1}{2}\right)\right)}_{s_1}, \underbrace{\ldots\ldots\ldots\ldots \ldots\ldots\ldots\ldots}_{s_2}, \ldots\ldots\ldots, \underbrace{m, \ldots\ldots, \left(m + \left(\frac{p}{m} - 1\right)\left(\frac{r+1}{2}\right)\right)}_{s_m} \right\}.$$

This is indeed a permutation of the integers from 1 to p. We then add edges between pairs of vertices that correspond to pairs of consecutive elements of this long sequence as follows: we add an edge between the first and the second element, then an edge between the third and the fourth element, then an edge between the fifth and the sixth, etc. There are p elements and p is odd. Therefore, the last edge that will be added is the one between the $(p-2)$th element and the $(p-1)$th element, and the vertex corresponding to the pth element will not receive a new edge in this procedure. In total, $(p-1)/2$ edges will be added and these edges are exactly the elements of E'. It is clear from the way the edges in E' are defined that exactly $p-1$ vertices of $R'_{p,r}$ have degree r and one vertex of it has degree $r-1$. Keeping in mind that each sequence s_ℓ has p/m elements, which is an odd number, Fig. 2 illustrates the pairs that correspond to the edges in E'.

To give an example, let us construct $R'_{15,11}$. We first construct $R_{15,10}$ as shown in Fig. 1, the edge set of which we call E. We have $m = \gcd(p, r+1) = \gcd(15, 12) = 3$. We form the sequences

$$s_1 = \{1, 7, 13, 4, 10\}, s_2 = \{2, 8, 14, 5, 11\}, s_3 = \{3, 9, 15, 6, 12\},$$
$$s = \{1, 7, 13, 4, 10, 2, 8, 14, 5, 11, 3, 9, 15, 6, 12\}.$$

We add edges on the graph $R_{15,10}$ between the vertices $\{v_1, v_7\}$, $\{v_{13}, v_4\}$, $\{v_{10}, v_2\}$, $\{v_8, v_{14}\}$, $\{v_5, v_{11}\}$, $\{v_3, v_9\}$, and $\{v_{15}, v_6\}$. This is the edge set E' and Fig. 3 shows the graph $R'_{15,11}$ where each vertex has degree 11, except one, v_{12}, which is of degree 10. The edges of E' are shown in bold.

Observation 7. *$R'_{i+1,i-1}$ is isomorphic to K'_i, for even i.*

Observation 8. *$E \cap E' = \emptyset$.*

Lemma 9. *$R'_{p,r}$ is claw-free.*

Proof. The addition of the edges in E' to $R_{p,r-1}$ creates two different types of vertices since there are two types of edges in E'. Let us denote by U_1 the set of vertices that appear in pairs in the same sequence s_ℓ for some ℓ between 1 and m, and by U_2 the set of vertices that appear in a pair consisting of the last element of a sequence s_ℓ and the first element of the next

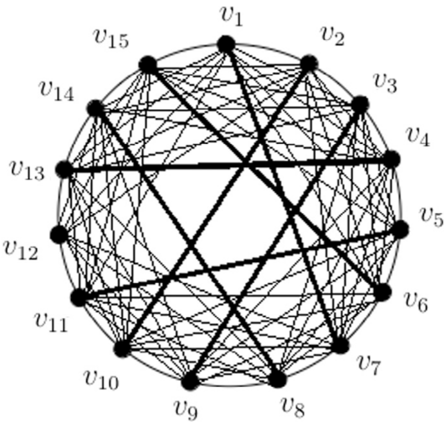

Fig. 3. The graph $R'_{15,11}$.

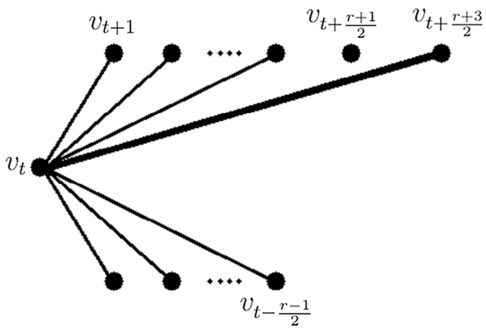

Fig. 4. Vertices in U_2 are not claw-centers.

sequence $s_{\ell+1}$. We also note that there is exactly one vertex in $R'_{p,r}$ which is neither in U_1 nor in U_2 since it is not incident to an edge in E'. We will prove that the addition of the edges in E' to $R_{p,r-1}$ does not create claws, by showing that no vertex in $R'_{p,r}$ is a claw-center.

In $R_{p,r-1}$, every vertex is adjacent to the $\frac{r-1}{2}$ closest vertices on the circle on each side. Every vertex u in U_1 gets an additional neighbor at distance $\frac{r+1}{2}$ on one side of the circle. Since r is odd, this additional new neighbor w of u is next to a furthest-away previous neighbor on that side. Now, observe that in $R_{p,r-1}$, the vertices between u and w on that side of the circle form a clique. Thus the new neighborhood of u on that side of the circle forms a clique. Consequently, the right and left neighbors of the vertices in U_1 form two cliques, and therefore the vertices in U_1 are not claw-centers.

The vertices in U_2 are adjacent to their closest $(\frac{r+3}{2})$th neighbor on the cycle instead of the $(\frac{r+1}{2})$th, hence there is a gap of one vertex in the neighborhood on the one side. Note that U_2 might be empty (this occurs when $m = 1$). Let us argue that the vertices in U_2 are not claw-centers. In Fig. 4, where the indices of the vertices should be considered in modulo p, v_t is a vertex of U_2. The vertices v_t and $v_{t+\frac{r+1}{2}}$ have indices in the same sequence and v_t is adjacent to $v_{t+\frac{r+3}{2}}$ whose index belongs to the next sequence. By definition $v_{t+\frac{r+3}{2}}$ is also a vertex of U_2. Furthermore, the vertices v_{t+1}, $v_{t+\frac{r+3}{2}}$, and $v_{t-\frac{r-1}{2}}$ have all indices in the same sequence. Now, the important observation is that each sequence has at most one vertex adjacent to a vertex from another sequence. Hence, v_{t+1} is a vertex of U_1 since $v_{t+\frac{r+3}{2}} \in U_2$. As an element of U_1, v_{t+1} has one of two possible neighbors: $v_{t+\frac{r+3}{2}}$ and $v_{t-\frac{r-1}{2}}$. Since $v_{t+\frac{r+3}{2}}$ is adjacent to v_t, completing its degree to r, we conclude that v_{t+1} is adjacent to $v_{t-\frac{r-1}{2}}$. Now, since v_{t+1} is adjacent to $v_{t-\frac{r-1}{2}}$, the neighbors $\{v_{t+1}, v_{t-1}, v_{t-2}, \ldots, v_{t-\frac{r-1}{2}}\}$ of v_t form a clique of size $\frac{r+1}{2}$ and the other neighbors of v_t form a clique of size $\frac{r-1}{2}$. By Observation 1, we conclude that no vertex of U_2 is a claw-center.

There is a single vertex remaining outside of U_1 and U_2, namely the last vertex in the long sequence. This vertex is not incident to any edge of E', and thus it is not a claw-center, since $R_{p,r-1}$ is claw-free. Since no vertex of $R'_{p,r}$ is a claw-center, we conclude that it is claw-free. □

Back to our example, for the graph $R'_{15,11}$, $U_1 = \{v_1, v_7, v_{13}, v_4, v_8, v_{14}, v_5, v_{11}, v_3, v_9, v_{15}, v_6, v_{12}\}$ and $U_2 = \{v_{10}, v_2\}$.

4.3. The main result

We are ready to present our main result.

Theorem 10. *If $i < 2j$, then an edge-extremal graph in $\mathcal{M}_{\mathcal{CF}}(i, j)$ can be obtained by taking*

$$\begin{cases} q-1 \text{ copies of } K_i \text{ and } 1 \text{ copy of } R_{i+2r,i-1} & \text{if } i \text{ is odd} \\ q-1 \text{ copies of } R'_{i+1,i-1} \text{ and } 1 \text{ copy of } R'_{i+2r+1,i-1} & \text{if } i \text{ is even,} \end{cases}$$

where q and r are respectively the quotient and the remainder of the division of $j-1$ by $\lceil \frac{i-1}{2} \rceil$.

The number of edges in such a graph is $(i-1)(j-1) + \lfloor \frac{i-1}{2} \rfloor \left\lfloor \frac{j-1}{\lceil \frac{i-1}{2} \rceil} \right\rfloor$.

Proof. If $i < 2j$, then $\lceil \frac{i-1}{2} \rceil \le j-1$, and thus $q \ge 1$. Hence the phrase "$q-1$ copies" is always meaningful. We have to show that the suggested graphs belong to $\mathcal{M}_{\mathcal{CF}}(i, j)$ and they are edge-extremal.

- *The suggested graphs are in $\mathcal{M}_{\mathcal{CF}}(i, j)$*
  Showing that they are in $\mathcal{M}_{\mathcal{CF}}(i, j)$ is equivalent to show that they are in $\mathcal{M}_{\mathcal{GEN}}(i, j)$ and claw-free.
  When i is odd, $\Delta(K_i) = i-1$, $\nu(K_i) = \frac{i-1}{2}$, $\Delta(R_{i+2r,i-1}) = i-1$, and $\nu(R_{i+2r,i-1}) = \frac{i+2r-1}{2}$. Therefore the maximum degree of the graph suggested in our theorem is $i-1 < i$ and its matching number is $(q-1)(\frac{i-1}{2}) + \frac{i+2r-1}{2} = q(\frac{i-1}{2}) + r = j-1 < j$. Therefore this graph is in $\mathcal{M}_{\mathcal{GEN}}(i, j)$. Also, it is claw-free, since complete graphs are claw-free, and we showed in the previous section that $R_{i+2r,i-1}$ is claw-free. We conclude that the graph is in $\mathcal{M}_{\mathcal{CF}}(i, j)$.
  When i is even, $\Delta(R'_{i+1,i-1}) = i-1$, $\nu(R'_{i+1,i-1}) = \frac{i}{2}$, $\Delta(R'_{i+2r+1,i-1}) = i-1$, and $\nu(R'_{i+2r+1,i-1}) = \frac{i+2r}{2}$. Therefore the maximum degree of the graph suggested in the statement of our theorem is $i-1 < i$ and its matching number is $(q-1)\frac{i}{2} + \frac{i+2r}{2} = q\frac{i}{2} + r = j-1 < j$. Hence it is in $\mathcal{M}_{\mathcal{GEN}}(i, j)$. Furthermore, it is claw-free because $R'_{p,r}$, where p and r are odd and $r+2 \le p \le 2r+1$, is claw-free by Lemma 9. These conditions hold for both $R'_{i+1,i-1}$ and $R'_{i+2r+1,i-1}$, since in the latter we have $r < \frac{i}{2}$ due to the division rule.
- *The suggested graphs are edge-extremal in $\mathcal{M}_{\mathcal{CF}}(i, j)$.*
  We will show that the suggested graphs have the same number of edges as the edge-extremal graphs in $\mathcal{M}_{\mathcal{GEN}}(i, j)$.
  When i is odd, the graph that we suggest as an edge-extremal graph in $\mathcal{M}_{\mathcal{CF}}(i, j)$ is the disjoint union of $q-1$ copies of K_i and one copy of $R_{i+2r,i-1}$. Keeping in mind that $q(\frac{i-1}{2}) + r = j-1$ and $q = \left\lfloor \frac{j-1}{\frac{i-1}{2}} \right\rfloor$ for odd i, the number of edges of this graph is
  $$\frac{(i+2r)(i-1)}{2} + (q-1)\frac{i(i-1)}{2} = (i-1)(j-1) + \left(\frac{i-1}{2}\right) \left\lfloor \frac{j-1}{\frac{i-1}{2}} \right\rfloor.$$
  Observe that this is exactly the bound of Theorem 2 for odd i.
  When i is even, the graph that we suggest as an edge-extremal graph in $\mathcal{M}_{\mathcal{CF}}(i, j)$ is the disjoint union of $q-1$ copies of $R'_{i+1,i-1}$ and one $R'_{i+2r+1,i-1}$. Keeping in mind that $q(\frac{i}{2}) + r = j-1$ and $q = \left\lfloor \frac{j-1}{\frac{i}{2}} \right\rfloor$ for even i, number of edges of this graph is
  $$\frac{(i+2r)(i-1)+(i-2)}{2} + (q-1)\frac{i(i-1)+(i-2)}{2} = (i-1)(j-1) + \left(\frac{i}{2} - 1\right) \left\lfloor \frac{j-1}{\frac{i}{2}} \right\rfloor.$$
  Observe that this is exactly the bound of Theorem 2 for even i.
  Thus the graphs in the statement of our theorem have the same number of edges as edge-extremal graphs in $\mathcal{M}_{\mathcal{GEN}}(i, j)$. Consequently, they are edge-extremal in $\mathcal{M}_{\mathcal{CF}}(i, j)$. □

The findings of this paper can be summarized in the following corollary of Theorems 4 and 10.

Corollary 11. *Let $i, j \ge 2$ be two given integers.*
If $i \ge 2j$, then the unique edge-extremal graph in $\mathcal{M}_{\mathcal{CF}}(i, j)$ has $(2j-1)(j-1)$ edges.
If $i < 2j$, then an edge-extremal graph in $\mathcal{M}_{\mathcal{CF}}(i, j)$ has $(i-1)(j-1) + \lfloor \frac{i-1}{2} \rfloor \left\lfloor \frac{j-1}{\lceil \frac{i-1}{2} \rceil} \right\rfloor$ edges.

5. Concluding remarks

Some questions remain open that could be interesting to investigate further. For example, what is the maximum number of edges in a triangle-free graph whose maximum degree and matching number are bounded? As edge-extremal graphs both in the general case and in the claw-free case contain triangles, the answer to this question seems intriguing.

Another interesting question is the maximum number of edges in a chordal or interval graph whose maximum degree and matching number are bounded. Note that the graph K_i' is not chordal. Proper interval graphs, which are exactly the claw-free interval graphs [7], have been studied in this respect by Måland [8]. When we combine his results with ours, we see that the interval condition on claw-free graphs significantly reduces the number of edges of an edge-extremal graph. For instance, an edge-extremal graph in $\mathcal{M}_{\mathcal{CF}}(10, 7)$ has 58 edges, whereas an edge-extremal graph in $\mathcal{M}_{\mathcal{PI}}(10, 7)$ has 48 edges, where $\mathcal{PI}$ denotes the class of proper interval graphs.

Finally, we observe that edge-extremal graphs are mostly disconnected. Adding connectivity constraint might lead to a significant decrease in the number of edges of edge-extremal graphs.

Acknowledgment

This work was supported by the Research Council of Norway through grant number 249994. The first author was in addition supported by 213M620 TUBITAK Research Project.

References

[1] N. Balachandran, N. Khare, Graphs with restricted valency and matching number, Discrete Math. 309 (2009) 4176–4180.
[2] B. Bollobás, Extremal Graph Theory, Dover Publications, New York, 2004.
[3] G. Chartrand, P. Zhang, Chromatic Graph Theory, Chapman & Hall/CRC Press, Boca Raton, FL, 2009.
[4] V. Chvátal, D. Hanson, Degrees and matchings, J. Combin. Theory Ser. 20 (1976) 128–138.
[5] P. Erdős, R. Rado, Intersection theorems for systems of sets, J. Lond. Math. Soc. 35 (1960) 85–90.
[6] P. Erdős, A.H. Stone, On the structure of linear graphs, Bull. Amer. Math. Soc. (N.S.) 52 (1946) 1087–1091.
[7] F. Gardi, The Roberts characterization of proper and unit interval graphs, Discrete Math. 307 (2007) 2906–2908.
[8] E. Maland, Maximum Number of Edges in Graph Classes Under Degree and Matching Constraints (Master's thesis), University of Bergen, Norway, 2015.
[9] M. Simonovits, Extremal Graph Theory, Selected Topics in Graph Theory. II, Academic Press, London, New York, San Francisco, 1983.
[10] D.P. Sumner, Graphs with 1-factors, Proc. Amer. Math. Soc. 42 (1974) 8–12.
[11] P. Turán, On an extremal problem in graph theory, Mat. Fiz. Lapok 48 (1941) 436–452.